Introduction to Non-Commutative Geometry — by Examples

Kalyan B. Sinha

J. N. Centre for Adv. Sc. Research
and
Indian Institute of Science,
Bangalore.

Algebra A of functions on a compact topological space X has the following properties:

(i) identity fcn $e(x) = 1 \ \forall x$ belongs to this.

(ii) the norm (a measure of 'length')

$$\|f\| = \sup_{x \in X} |f(x)|$$

(iii) A is commutative: $fg(x) \equiv f(x)g(x) = (gf)(x)$.

(iv) $*: f \to \bar{f}$ where $\bar{f}(x) = \overline{f(x)}$ is an isometric "involution" (trivial here)

$\|f\| = \|f^*\|$ and $(fg)^* = g^* f^*$.

(v) A satisfies the C^*-property:

$$\|f^* f\| = \|f\|^2 \equiv \sup|f(x)|^2.$$

Such an A is a commutative C^*-alg.

The converse is also true (Gelfand-Naimark): Every commutative C^*

algebra with identity is isometrically isomorphic (practically indistinguishable) from above such function algebra.

Q: (i) can one abstract topological (e.g. connectivity, homotopy etc.) properties of X ~~fo~~ to the algebraic properties of A ? or similarly for geometric properties (volume, curvature) of X ?

(ii) If we now drop the "commutativity" part, then can one compute these geometric quantities or topological invariants, though there are no "functions" on an "underlying space X" ?

Weyl C^*-algebra : Non-Commutative Plane

In the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^d)$, for $\alpha, \beta \in \mathbb{R}^d$ consider two unitaries:

$$(U_\alpha f)(t) = f(t+\alpha),$$

$$(V_\beta f)(t) = e^{it\beta} f(t) \quad \text{for } t \in \mathbb{R}^d.$$

Then

$$\left.\begin{aligned} U_\alpha V_\beta &= e^{i\alpha.\beta} V_\beta U_\alpha, \\ U_\alpha U_{\alpha'} &= U_{\alpha'} U_\alpha = U_{\alpha+\alpha'}, \\ V_\beta V_{\beta'} &= V_{\beta'} V_\beta = V_{\beta+\beta'} \end{aligned}\right\} \text{Weyl Commutation Relations}$$

This can be written another way: Set for $x \equiv (\alpha, \beta) \in \mathbb{R}^{2d}$

$$W_x \equiv U_\alpha V_\beta \, e^{-i/2\, \alpha.\beta}.$$

Then

$$W_x W_y = W_{x+y}\, e^{i/2\, \phi(x,y)},$$

where ϕ is a symplectic form on $\mathbb{R}^{2d}$ given by $\phi(x,y) = \alpha_1\beta_2 - \alpha_2\beta_1$ where $x = (\alpha_1, \beta_1)$, $y = (\alpha_2, \beta_2)$. This is called the Segal Commutation relations and these two relations are equivalent.

Next, for a function f on $\mathbb{R}^{2d}$ (in general $\mathbb{C}$-valued) such that $\hat{f}$ (its Fourier transform) is in $L^1(\mathbb{R}^{2d})$, define

$$b(f) \equiv \int_{\mathbb{R}^{2d}} \hat{f}(x)\, W_x\, dx .$$

Then it is clear that all such $b(f)$'s are bounded linear operators in $\mathcal{H}_0$ and one can verify

$$b(f)\,b(g) = b(f \odot g),$$

where $\widehat{(f \odot g)}(x) = \int_{\mathbb{R}^{2d}} \hat{f}(x-x')\,\hat{g}(x')\,e^{i/2\,p(x,x')}\,dx'$,

a "slightly twisted convolution";

$b(f)^*$ (the adjoint in $\mathcal{H}_0$)

$$= b(f^{\natural}), \quad \text{with}$$

$$f^{\natural}(x) = \bar{f}(-x).$$

<u>Notation:</u>

$\mathcal{A}^{\infty} \equiv$ *-algebra generated by $\{ b(f) \mid f \in C_c^{\infty}(\mathbb{R}^{2d}) \}$

and

$\mathcal{A} = C^*$-algebra generated by $\{ b(f) \mid f \in C_c^\infty(\mathbb{R}^{2d}) \}$.

$\mathcal{A} \equiv$ Weyl C^*-algebra, $\mathcal{A}^\infty =$ smooth Weyl Algebra.

Remarks: (1) $\mathcal{A}^\infty$ and $\mathcal{A}$ are <u>not</u> commutative since

$$\widehat{g \odot f}(x) = \int \hat{f}(x-x')\,\hat{g}(x')\, e^{-i/2\, p(x,x')}\, dx'$$

$$\neq \widehat{f \odot g}(x).$$

(2) Define a positive linear functional τ on $\mathcal{A}^\infty$ by

$$\tau(b(f)) \equiv (2\pi)^d \hat{f}(0) = \int f(x)\, dx.$$

This defines a faithful trace on $\mathcal{A}^\infty$ (it's a trace because

$$\widehat{g \odot f}(0) = \widehat{f \odot g}(0).$$

Next we draw up a comparison table of the classical (commutative) algebra of functions under pointwise multiplication and the Weyl algebra, that has just been constructed.

classical	Weyl
(1) $A^\infty_{cl} \equiv C^\infty_c(\mathbb{R}^{2d})$ under ptwise mult. & complex conjugation as $*$-op^n. $A_{cl} \equiv C_0(\mathbb{R}^{2d})$. COMMUTATIVE	A^∞ and A respectively as defined above. non-COMMUTATIVE
(2) The translation group $\mathbb{R}^{2d}$ acts naturally on A^∞_{cl}, the derivations associated with it give the derivative op^n $(\partial_j f)(x) = \frac{\partial f}{\partial x_j}(x)$	A similar action: $\varphi_\alpha(b(f)) \equiv b(f_\alpha)$, $\hat{f}_\alpha(x) = e^{i\alpha \cdot x}\hat{f}(x)$ gives derivation: $\delta_j(b(f)) = b(\partial_j f)$.

classical	Weyl
) $\tau(f) = \int f(x)\,dx$ gives a tr linear f'cnal on A_{cl}^{∞} which is a trace obviously since A_{cl}^{∞} is commutative.	Same expression gives a trace. Remark: $L^2(A^{\infty}, \tau)$, the non-commutative L^2-space in which A^{∞} is represented by left multiplication: the map $C_c^{\infty}(\mathbb{R}^{2d}) \ni f \longmapsto b(f) \in A^{\infty} \subseteq L^2(A^{\infty}, \tau)$ extends as an unitary isomorphism betn $L^2(\mathbb{R}^{2d}) \cong L^2(A^{\infty}, \tau)$.
(4) The Laplacian here $\Delta \equiv -\sum_{j=1}^{2d} \partial_j^2$ and acting on A_{cl}^{∞}	Again by analogy the natural 'Laplacian or Lindbladian' $\mathcal{L}(b(f)) \equiv \frac{1}{2} b(\Delta f)$, acting on A^{∞}

Remarks:

If we now go back to the defining Weyl-Segal relations repn of the Weyl-Segal relations in $\mathcal{H}_{\gamma} = L^2(\mathbb{R}^d)$, then

$$b(f) = \int_{\alpha\in\mathbb{R}^d,\ \beta\in\mathbb{R}^d} \hat{f}(\alpha,\beta)\, U_\alpha V_\beta\, e^{-i/2\,\alpha.\beta}\, d\alpha\, d\beta,$$

and

$$\varphi_{(s,t)}(b(f)) \equiv b(f_{(s,t)})$$ where

$$\hat{f}_{(s,t)}(\alpha,\beta) = e^{is\alpha}e^{it\beta}\hat{f}(\alpha,\beta).$$

Thus $\delta_j(b(f)) = -i\left[Q_j, b(f)\right]$ and ~~for j=1,2,…d~~,

$$\delta_{j+d}(b(f)) = -i\left[P_j, b(f)\right] \text{ for } j = \cancel{d+1,\ldots 2d}, = 1,2,\ldots d$$

where $U_\alpha = e^{i\sum_{j=1}^{d} P_j.\alpha_j}$ and

$V_\beta = e^{i\sum_{j=1}^{d} Q_j \beta_j}$ with $\{Q_j, P_j\}_{j=1}^{d}$ being the so-called position & momentum operators (multiplication by x_j and the generator of translation in jth direction in $L^2(\mathbb{R}^d)$ respectively) in Quantum Mechanics.

Though $[P_j, Q_k] = -i\,\delta_{jk}$, i.e. they do not commute in general,

$$\delta_j \delta_k (b(f)) = -[q_j, [p_k, b(f)]]$$
$$= -[p_k, [q_j, b(f)]] = \delta_k \delta_j (b(f)),$$

and

Jacobi Identity

For convenience of comparison:

$$\widehat{f\cdot g}(y) = (2\pi)^{-d}\int f(x)\,g(x)\,e^{-iy.x}\,dx$$

$$= (2\pi)^{-2d}\int\!\!\int \hat{f}(k)\,e^{ik.x}\,dk \int \hat{g}(k')\,e^{ik'.x}\,dk'\; e^{-iy.x}\,dx$$

$$\cancel{= (2\pi)^{-d}\int \hat{f}(k)\,\hat{g}(y-k)\,dk = \widehat{g\cdot f}(y)},$$

$$= \int \hat{f}(y-k)\,\hat{g}(k)\,dk$$

whereas

$$\widehat{f\odot g}(y) = \int \hat{f}(y-k)\,\hat{g}(k)\,e^{i/2\,p(y,k)}\,dk$$

$$\neq \widehat{g\odot f}(y).$$

In $\mathbb{R}^{2d} \simeq \mathbb{R}^d \times \mathbb{R}^d$, the translations along the two distinguished $\mathbb{R}^d$-directions commute classically. But in this model, they do not, leading to the term "non-commutative plane"

Classical	Weyl
	$\mathcal{L}_0 = -\frac{1}{2}\left\{ \sum_{j=1}^{d} \left([P_j, [P_j, \cdot\,]] + [Q_j, [Q_j, \cdot\,]] \right) \right\}$. This is the non comm. Laplacian on the <u>non-commutative</u> 2d-plane.
(5) Heat Semigroup $T_t = \exp(\frac{t}{2}\Delta)$ on $\mathcal{A}^\infty_{cl}$	"Heat Semigroup" $T_t = \exp(t\mathcal{L}_0)$ on $\mathcal{A}^\infty$

Classical	Weyl
(6) $\lim_{t\to 0+} \left[\mathrm{Tr}\left(M_f T_t\right)\right] t^{2d/2} = \int_{\mathbb{R}^{2d}} f(x)\,dx$, $f \in A^\infty_{cl}$, M_f the bounded op. of mult. by f	$\lim_{t\to 0+} \left[\mathrm{Tr}\left(b(f)\, T_t\right)\right] t^{2d/2} = \int_{\mathbb{R}^{2d}} f(x)\,dx$, for $b(f) \in A^\infty$.

Remark: In fact in (6) above

$$\mathrm{Tr}\left(M_f T_t\right) = t^{-2d/2} \int_{\mathbb{R}^{2d}} f(x)\,dx$$

in both cases, showing that these are 'flat' spaces.

The group $\mathbb{R}^{2d}$ acts as:

$$\Phi_\alpha(b(f)) = b(f_\alpha) \text{ with}$$

$\hat{f}_\alpha(x) = e^{i\alpha x}\hat{f}(x)$. The 'Laplacian' is $-\sum_{j=1}^{2d} \delta_j^2$ and the corresponding Heat semigroup T_t is given by:

$$T_t(b(f)) = \int_{\mathbb{R}^{2d}} e^{-t/2\, x^2}\hat{f}(x)\, W_x\, dx.$$

One can easily verify in this (non commutative) case that

$$T_t(b(f_\alpha)) = \Phi_\alpha(T_t(b(f))) \text{ and}$$

$$\tau(T_t(b(f)^*)\, b(u))$$

$$= \int_{\mathbb{R}^{2d}} e^{-\frac{tx^2}{2}} \overline{\hat{f}}(x)\hat{u}(x)\, dx$$

$$= \tau(b(f)^* T_t(b(u))) \text{ showing}$$

covariance & symmetry.

Example 2: Non-Commutative Torus

$A = C^*\{U, V$ unitaries with relation: $UV = VU e^{2\pi i\theta}$, θ irrational in $[0,1)$.

Remark: (1) In classical torus $\mathbb{T}^2$, U and V can be taken to be the op. of multiplication by z_1 and z_2 resp. in $L^2(\mathbb{T}^2)$ with $|z_1| = |z_2| = 1$. It can be seen that if θ is rational $= m/n$, say, then $U^n V^n = V^n U^n$ reverting back to the classical structure.

(2) The norm in $\mathcal{A}$ is given by $\|x\| = \sup_{\pi} \|\pi(x)\|$ where π is the family of all $*$ representations of $\mathcal{A}$.

There is a natural action of the group $\mathbb{T}^2$ on A by

$$\alpha_{z_1, z_2}(U, V) = (z_1 U, z_2 V)$$
$$|z_1| = |z_2| = 1,$$

and the associated derivations are:

$$d_1(U) = U \ , \quad d_1(V) = 0$$

$$d_2(U) = 0 \ , \quad d_2(V) = V .$$

von Neumann algebras. The simplest example is that of the "irrational rotation algebra" (or non-commutative torus) (Jour. Op. Theory. 49, 2003).

Consider the non-commutative torus $\mathcal{A}_\Theta$ generated by unitaries subject to the relation :

$$UV = VU \exp(2\pi i \Theta) \equiv VU\lambda;$$

(i) where Θ is an irrational number is [0, 1]. There is a natural action α of the compact group T^2 on $\mathcal{A}_\Theta$, whose generators give the 2 cononical derivations :

$$d_1(U) = U,\ d_1(V) = 0$$

$$d_2(U) = 0,\ d_2(V) = V.$$

If we set $\mathcal{A}_\Theta^\infty = \Big\{a \in \mathcal{A}_\Theta = \Big| \sum_{m,n,\in \mathcal{Z}} a_{mn} U^m V^n$, $\sup_{m,n} |m|^k \, |n|^l \, |a_{mn}| \ \leq \ c$, for all $k, l \ \in \ \mathcal{N}\}$, then d_1 and d_2 are well-defined on $\mathcal{A}_\Theta^\infty$. Also there is a unique faithful trace τ on $\mathcal{A}_\Theta^\infty$ given as :

$$\tau(\sum a_{mn} U^m V^n) = a_{00}.$$

The one can consider $h = L^2(\mathcal{A}_\Theta, \tau)$ and note that d_1 and d_2 defined on their natural domains are self-adjoint. For example, $Dom(d_1)$ $= \ \Big\{\sum a_{mn} U^m V^n \Big| \sum (1 + |m|^2) \, |a_{mn}|^2 < \infty\Big\}$. It is clear that for $r \in \mathcal{A}_\Theta$ (acting as left multiplication in h) $d_r \ \equiv \ [r, .]$ has the property $d_r^* = d_r *$.

A theorem of Bratteli, et al, (J. reine. angew.

math. 346, 1984) describes all the derivations of $\mathcal{A}_\Theta^\infty$ to itself : for a. a. Θ(Lebesgue), a derivation of such kind is of the form $c_1 d_1 + c_1 d_2 + [r, .]$

At this point, we can have two Laplacians : $\mathcal{L}_0 = -\frac{1}{2}(d_1^2 + d_2^2)$ and $\mathcal{L} = -\frac{1}{2}(\delta_1^2 + \delta_2^2)$ acting on $\mathcal{A}_\Theta^\infty$, where $\delta_1 = d_1 + d_{r_1}, \delta_2 = d_2 + d_{r_2}$ with $r_1, r_2 \in \mathcal{A}_\Theta^\infty$, and consider the geometric quantities coming out of an asymptotic analysis of the associated heat semigroups in the spirit of Weyl in the 1920's.

Weyl Asymptotics

For classical Riemannian manifold (M, g) of dim d, the heat semigroup T_t is an in integral operator in L^2(M, dvol) with smooth kernal $T_t(x, y)$ admitting an expansion :

$$T_t(x,y) = \sum_{j=0}^{\infty} u^{(j)}(x,y)t^{-\frac{d}{2}+j},$$

where vol(M) $=\int_M u^{(0)}(x.y)dvol(x)$

$$= \lim_{t\to 0+} t^{\frac{d}{2}} \int_{M'} T_t(x,x)dvol(x)$$

$$= \lim_{t\to 0+} t^{\frac{d}{2}} T_r(T_t),$$

where T_r is the trace in L^2 $(M, dvol)$. Similarly the integrated scalar curvature s is given as $s = \frac{1}{6}\lim_{t\to 0+} t^{\frac{d}{2}-1}[T_r(T_t) - t^{-\frac{d}{2}}vol(M)]$.

By analogy, we define for $\mathcal{A}_\Theta$, volume $v(\mathcal{A}_\Theta) = \lim_{t\to 0+} t^{\frac{d}{2}} T_r(T_t)$ and

$$6s(\mathcal{A}_\Theta) = \lim_{t\to 0+} t^{\frac{d}{2}-1}[T_r T_t - t^{-\frac{d}{2}}v(\mathcal{A}_\Theta)],$$

where T_t can be either of the two dynamical semigroups on $\mathcal{A}_\Theta$ with generative $\mathcal{L}_0$ or $\mathcal{L}$, and T_r is the trace in $L^2(\mathcal{A}_\Theta, \tau)$.

Theorem 1 :

(i) $\mathcal{L}_0$ is a negative self-adjoint operator in h with compact resolvent, infact
$(\mathcal{L}_0 - z)^{-1} \in \mathcal{B}_p(h), p > 1$ and $z \in \rho(\mathcal{L}_0)$.

(ii) If $r_1, r_2 \in \mathcal{A}_\Theta^\infty$, then $\mathcal{L} = \mathcal{L}_0 + B + A$,
$B = -\frac{1}{2}(d_{r_1}^2 + d_{r_2}^2) + d_{d_1(r_1)} + d_{d_2(r_2)}$,
$A = -d_{r_1}d_1 - d_{r_2}d_2$, and A is compact relative to $\mathcal{L}_0$. Also $\mathcal{L}$ is self-adjoint on $\mathcal{D}(\mathcal{L}_0)$ with compact resolvent.

(iii) $(\mathcal{L} - z)^{-1} - (\mathcal{L} - z)^{-1}$ is trace class for $z \neq 0$ or $z > 0$.

The next theorem compares the geometric quantities associated with the two distinct Laplacians as given earlier.

Theorem 2:

(i) The volume $V(d = 2)$ as defined above is invariant under perturbation $\mathcal{L}_0$ to $R\mathcal{L}$.

(ii) The integrated scalar curvature s with $r \in \mathcal{A}_\Theta^\infty$ is however, not in general invariant under the perturnation.

Sketch of proof :

We use the formule that

$$e^{t\mathcal{L}} - e^{t\mathcal{L}_0} = -\int_0^t e^{(t-s)\mathcal{L}_0}(\mathcal{L} - \mathcal{L}_0)e^{s\mathcal{L}}ds$$

and iterate it a few items to show that

$$T_r(e^{t\mathcal{L}} - e^{t\mathcal{L}}) = 0(t^{-\frac{1}{2}}) + 0(1) + 0(1)$$

as $t \to 0+$, thereby giving the result for d $= 2$. A similar calculation can be made for any d.

By a similar kind of estimation, we show that (for d $= 2$)

$$6\left[s(\mathcal{L}) - s(\mathcal{L}_0)\right] = -\lim_{t\to 0+} tT_r\left((\mathcal{L} - \mathcal{L}_0)e^{t\mathcal{L}_0}\right)$$

As before, $\mathcal{L} - \mathcal{L}_0 = B + A$, and we claim that while $T_r(Ae^{t\mathcal{L}_0}) = 0 \forall t > 0$, $T_r(Be^{t\mathcal{L}_0}) \neq 0$ in general. For the first note that

$$T_r(d_r d_1 e^{t\mathcal{L}_0}) = \sum_{m,n} \tau(< U^m V^n, d_r d_1 e^{t\mathcal{L}_0}(U^m V^n))$$

$$= \sum_{m,n} m e^{\frac{-t(m^2+n^2)}{2}\tau(V^{-m}U^{-n}rU^mV^n - r) = 0}$$

For the second one, let us consider the case $r_1 = r = U + U^{-1}$, and $r_2 = 0$. Then simple algebra tells us that

$$T_r B e^{t\mathcal{L}_0} = \frac{1}{12} \sum e^{-\frac{t}{2}(m^2+n^2)} \tau(< U^m V^n d_{r_1}^2 U^m V^n >).$$

$$= Const.(\sum_{m+1}^{\infty} e^{-\frac{m^2 t}{2}} + 1)(\sum_{n+1}^{\infty} (\sin^2 \pi \Theta n) e^{-\frac{n^2 t}{2}})$$

For $0 < t < 2$,

$$\sqrt{t}\sum^{\infty} e^{-\frac{m^2t}{2}} \geq \sqrt{t}\sum_{m=1}^{[\sqrt{\frac{2}{t}}]} e^{-\frac{m^2t}{2}} \geq e^{-1}(\sqrt{2}-\sqrt{t}),$$

and

$$\sqrt{t}\sum_{n=1}^{\infty}(\sin^2 n\pi\Theta)e^{-\frac{n^2t}{2}} \geq \sqrt{t}\sum_{n=1}^{[\sqrt{\frac{2}{t}}]}(\sin^2 n\pi\Theta)e^{-\frac{n^2t}{2}}$$

$$\geq e^{-1}(\sqrt{2}-\sqrt{t})\sum_{n=1}^{[\sqrt{\frac{2}{t}}]}[\sqrt{\frac{2}{t}}]^{-1}\sin^2\pi(n\Theta-[n\Theta])$$

$$= e^{-1}(\sqrt{2}-\sqrt{t})\mathcal{E}(\sin^2\pi X_t)$$

Where for each t, X_t is a [0, 1] - valued random variable with Prob $(X_t = k\Theta - [k\Theta]) = [\sqrt{\frac{2}{t}}]^{-1}$ for k = 1, 2, ..., $[\sqrt{\frac{2}{t}}]$ and $\mathcal{E}$ is the associated expectation. Since Θ is irrational, it is known that as $t \to 0+$, $X_t \to_{weakly}$ a random variable with uniform distribution on [0, 1] and therefore

$$\lim_{t\to 0+} \sqrt{t}\sum_{n=1}^{\infty} \sin^2(\pi\Theta n)e^{-\frac{n^2t}{2}}$$

$$\geq \sqrt{2}e^{-1}\int_0^1 \sin^2 \pi x dx = \frac{1}{\sqrt{2}e} > 0.$$

For example,

$r_1 = r = V + V^{-1}$ and $r_2 = 0$,

$6\big(s(\mathcal{L}) - s(\mathcal{L}_0)\big) \geqslant \frac{1}{\sqrt{2}\,e}$.

Spectral Triple (Connes)

$(\mathcal{A}^\infty, \mathcal{H}, \mathcal{D})$,

$\mathcal{A}^\infty$ is a $*$-subalgebra, dense in the C^*-algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$, $\mathcal{H}$ a Hilbert space and D a self-adjoint operator in $\mathcal{H}$ with compact resolvent such that $[D, a] \in \mathcal{B}(\mathcal{H})$ $\forall a \in \mathcal{A}^\infty$. The triple is even if $\exists$ a $\mathbb{Z}_2$-grading Γ on $\mathcal{H}$ compatible with the triple, i.e. $\Gamma D + D\Gamma = 0$ & $\Gamma a = a\Gamma$. If such a grading does not exist, then the triple is odd. D is called the Dirac operator.

$\mathcal{A}$ a unital C^*-algebra. Universal <u>Differential algebra of forms</u> $\Omega\mathcal{A} = \oplus \Omega_k \mathcal{A}$ graded alg.: $\Omega^0\mathcal{A} = \mathcal{A}$, $\Omega'\mathcal{A}$ made up of symbols δa as $\mathcal{A}$-bi-module satisfying Leibnitz' rule etc. A k-form will look like

$$\left\{ \sum_{j=1}^{n} a_0^{(j)} \delta a_1^{(j)} \cdots \delta a_k^{(j)} \;\middle|\; n \in \mathbb{N},\ a_\ell^{(j)} \in \mathcal{A} \right\},$$ and $\delta^2 = 0$ and

$$(\delta(a))^* = -\delta(a^*) \Rightarrow \Omega^\cdot \text{ is a } *\text{-algebra.}$$

<u>Conne's differential forms</u>

Given a spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$, let π be a $*$-representation of $\Omega^\cdot\mathcal{A}$ in $\mathcal{B}(\mathcal{H})$ by $\pi(a_0 \delta(a_1)\delta(a_2) \cdots \delta(a_p)) = a_0 [D, a_1] \cdots [D, a_p]$, for $a_j \in \mathcal{A}^\infty$. Since D is self adjoint, one has

$$\pi(\delta(a)^*)^* = [D, a]^* = -[D, a^*] = -\pi(\delta(a^*)) = \pi(\delta(a)^*).$$

Try to define forms as image of $\pi(\Omega^\cdot\mathcal{A})$, but doesn't work since $\pi(\omega) = 0$ does not imply $\pi(\delta\omega) = 0$, i.e. there are "bad/junk" forms al has to be ~~quotion~~ divided out.

Let $J_0^{(k)} = \ker\left(\pi|_{\Omega^k}\right)$ and $J_0 \equiv \bigoplus J_0^{(k)}$. Then $J \equiv J_0 + \delta J_0$ is a graded 2-sided differential ideal of $\Omega^\cdot A$ and we set

$$\Omega_D A \equiv \Omega A / J \simeq \pi(\Omega A)/\pi(\delta J_0),$$

$\pi(J_0) = 0$ by definition. Also define $\Omega_D^p A = \Omega^p A / J^p$ and $d[\omega] = [\delta\omega]$ so that

$$d : \Omega_D^p A \longrightarrow \Omega_D^{p+1} A.$$

Example of spin manifold: (M, g) n-dim compact Riemannian spin manifold, $A = C(M)$, $A^\infty = C^\infty(M)$, $\mathcal{H} = L^2(M, S)$, sq-integrable sections of spinor-bundles on M of rank $2^{[n/2]}$; the metric g given by $g^{\mu\nu} = e^\mu_\alpha e^\nu_\beta \eta^{\alpha\beta}$, where $\{e_\alpha\}$ is ONB of the TM. Let $C(M)$ be the Clifford-bundle over M whose fiber at x is $\mathrm{Cliff}_{\mathbb{C}}(T_x^* M)$; the morphism γ: sections $\Gamma(M, C(M)) \mapsto \mathcal{B}(\mathcal{H})$ by $\gamma(dx^\mu) \equiv \gamma^\mu(x) = \gamma^\alpha e^\mu_\alpha$ so that $\gamma^\mu(x)\gamma^\nu(x) + \gamma^\nu(x)\gamma^\mu(x) = -2g^{\mu\nu}(x)$.

The Dirac operator is given as $D = \gamma(dx^\mu)\nabla_\mu^S$ with

$\underbrace{\nabla_\mu^S = \partial_\mu + \tfrac{1}{2}\omega_{\mu ab}\gamma^a\gamma^b}_{\text{Covariant Derivative}}$, $\omega = dx^\mu \omega_\mu$ is the Levi-Civita connection.

Dixmier Trace and dimension & volume.

The (usual/matrix) trace on $\mathcal{B}(\mathcal{H})$ is a map $\tau: \mathcal{B}_+(\mathcal{H}) \longmapsto [0,\infty]$ s.t. (i) normal: $A_n \uparrow A \Rightarrow \tau(A_n) \uparrow \tau(A)$ ~~Linear extended~~ and (ii) unitary invariant: $\tau(uAu^{-1}) = \tau(A)$ $\forall A \in \mathcal{B}_+(\mathcal{H})$, u unitary.

Then linearly extended so as to enjoy trace property $\tau(AB) = \tau(BA)$. All such operators A for which $A = A_1 - A_2 + i(A_3 - A_4)$ with $A_j \in \mathcal{B}_+(\mathcal{H})$ and $\tau(A_j) < \infty$ are called trace-class $\Rightarrow$ These form a Banach space and 2-sided ideal in $\mathcal{B}(\mathcal{H})$ and denoted $\mathcal{B}_1(\mathcal{H})$.

Let $\mu_j(A)$ be the singular values of a compact op. A (i.e. eigenvalues of $(A^*A)^{1/2}$), arranged in decreasing order, and set

$$\sigma_N(A) = \sum_{j=0}^{N-1} \mu_j(A) \qquad (\mu_0 \geq \mu_1 \geq \cdots)$$

We say that $A \in \mathcal{B}_{1,\infty}(\mathcal{H})$, the Dixmier class if $\|A\|_{1,\infty} = \sup_N \left(\frac{\sigma_N(A)}{\log N}\right) < \infty$ i.e. $\sigma_N(A) = O(\log N)$.

Like the matrix-trace, want to define a map Compact +ve $\ni T \longmapsto \lim_{N\to\infty} \frac{1}{\log N} \sigma_N(T)$ and extend it linearly. If the limit exists, then it is clearly unitary invariant and hence will extend as a trace. But the limit does NOT exist, in general, and a special kind of Banach limit has to be defined to preserve the trace property and to have uniqueness $\Rightarrow Tr_\omega(\cdot)$, Dixmier Trace.

Properties

(i) $T \in \mathcal{B}_{1,\infty,+} \Rightarrow 0 \leq Tr_\omega(T) < \infty$,

(ii) $T \in \mathcal{B}_{1,\infty}, S \in \mathcal{B} \Rightarrow ST \,\&\, TS \in \mathcal{B}_{1,\infty}$ and $Tr_\omega(ST) = Tr_\omega(TS)$

(iii) $T \in \mathcal{B}_1 \Rightarrow Tr_\omega(T) = 0$, (iv) $S \in \mathcal{B}_0$ (compact), $T \in \mathcal{B}_{1,\infty} \Rightarrow Tr_\omega(ST) = 0$.

Weyl $\Rightarrow vol(M) = \lim_{t\to 0+} t^{d/2}\, Tr(T_t)$ → Heat Semigroup on M

$= \lim t^{d/2} \sum_n e^{-t\mu_n(-\Delta)} = Tr_\omega((-\hat{\Delta})^{-d/2})$.

[HAR]DY-KARAMATA Theorem

Given a Laplacian (or Dirac)-operator on a M (or a NC-manifold), the dimension of $M \equiv d$ is the integer s.t.

$$Tr_\omega\left((-\hat{\Delta})^{-s/2}\right) = \begin{cases} \infty & \text{if } s < d \\ 0 & \text{if } s > d \\ \text{non-zero} & \text{if } s = d. \end{cases}$$

$\hat{\Delta} \equiv \Delta|_{\ker(\Delta)^\perp}$.